Solution to Assignment 2

Supplementary Problems

1. Let S be the sector bounded by the straight line $y = \tan\theta x$, the positive x-axis and the circle $x^2 + y^2 = r^2$. Show that its area is given by $\theta r^2/2$. To simply the calculation, you may assume $\theta \in (0, \pi/2]$.

   Solution. The line $y = \tan\alpha x$ and $x^2 + y^2 = r^2$ intersects at the point $(r\cos\alpha, r\sin\alpha)$. The sector S is described as

   $$S = \left\{(x, y) : (\tan\alpha)^{-1} y \le x \le \sqrt{r^2 - y^2},\ 0 \le y \le r\sin\alpha\right\} .$$

   Therefore, its area is given by

   $$\begin{aligned} \int_0^{r\sin\alpha} \int_{(\tan\alpha)^{-1}y}^{\sqrt{r^2-y^2}} 1\, dxdy &= \int_0^{r\sin\alpha} \left(\sqrt{r^2 - y^2} - \frac{1}{\tan\alpha} y\right) dy \\ &= \frac{1}{2}\alpha r^2 + \frac{r^2}{4}\sin 2\alpha - \frac{r^2}{4}\sin 2\alpha \\ &= \frac{1}{2}\alpha r^2 . \end{aligned}$$

2. Let f and g be continuous on the region D. Deduce the inequality

   $$2 \iint_D |fg|\, dA \le \alpha^2 \iint_D f^2\, dA + \frac{1}{\alpha^2} \iint_D g^2\, dA ,$$

   where α is a positive number. Hint: Use $(\alpha f(x) \pm \alpha^{-1} g(x))^2 \ge 0$.

   Solution. We have $(\alpha f(x, y) \pm \alpha^{-1} g(x, y))^2 \ge 0$, that is,

   $$\alpha^2 f^2(x, y) + \alpha^{-1} g^2(x, y) \ge 2|f(x, y)g(x, y)| .$$

   Integrating this inequality over D to get

   $$\alpha^2 \iint_D f^2\, dA + \alpha^{-2} \iint_D g^2\, dA \ge 2 \iint_D |fg|\, dA .$$

   Note that we have used linearity and positivity of the Riemann integral.

3. Setting as in (2), prove the Cauchy-Schwarz inequality:

   $$\iint_D |fg|\, dA \le \left(\iint_D f^2\, dA\right)^{1/2} \left(\iint_D g^2\, dA\right)^{1/2} .$$

   Solution. Choose

   $$\alpha^2 = \frac{\left(\iint_D g^2\, dA\right)^{1/2}}{\left(\iint_D f^2\, dA\right)^{1/2}} .$$

4. Let f be a non-negative continuous function on D and p a positive number. Show that

   $$m \le \left(\frac{1}{|D|} \iint_D f^p\, dA\right)^{1/p} \le M ,$$

where m and M are respectively the minimum and maximum of f and $|D|$ is the area of D.

Solution. From $m^p \le f(x,y)^p \le M^p$, we integrate to get

$$m^p|D| = \iint_D m^p \, dA \le \iint_D f^p \, dA \le= \iint_D M^p \, dA = M^p|D| \ ,$$

and the inequality follows.

Note. It shows $\left(|D|^{-1} \iint_D f^p \, dA\right)^{1/p}$ can also be used to describe some kind of average. The cases $p = 1, 2$ are most common.